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On the shot-noise limit of a thermal current

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Abstract

The noise power spectral density of a thermal current between two macroscopic dielectric bodies held at different temperatures and connected only at a quantum point contact is calculated. Assuming the thermal energy is carried only by phonons, we model the quantum point contact as a mechanical link, having a harmonic spring potential. In the weak coupling, or weak link limit, we find the thermal current analog of the well-known electronic shot-noise expression.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Just as Ohm's law, relating the electrical current to an applied potential, breaks down when the quantum mechanical aspects of the charge carriers becomes important, such as in the mesoscopic regime; Fourier's Law of heat conduction suffers a similar fate [1]. Mesoscopic phonon systems provide some of the best experimental setups to test the quantum nature of heat transport, such as the quantization of thermal conductance [1, 2]. Although, experimental demonstration of which lagged a decade behind that of its electronic counterpart [3].

Following this seminal work, nanomechanical systems have since attracted an increased interest, experimentally and theoretically, from such diverse areas as quantum computing [4] to promising research into detecting the quantum mechanical zero-point motion of a macroscopic object [5, 6]. Similar to the quantization of electrical conductance, where each channel of a one-dimensional conductor can contribute a quantum of electrical conductance, $e^2/2\pi\hbar$ per spin, in a one-dimensional dielectric each vibrational mode carries a quantum of thermal conductance given by $\pi k_B^2 T/6\hbar$. Of course one requirement to observe this quantization is a clean system with minimal scattering, i.e. ballistic transport. Within the Landauer–Büttiker formalism, this amounts to setting the transmission matrix to unity for each mode. The opposite limit of weak transmission or strong scattering can be equally interesting. For instance in a system of two conductors separated by a thin tunnel barrier, such as a scanning tunneling microscope (STM), the electrical conductance, associated with the tunneling current, is related to the product of the local density of states on each side of the barrier [7]. In [8] a thermal analog of an STM, i.e. a phonon scanning *thermal* microscope, was proposed, where the thermal conductance associated with the energy current between two macroscopic dielectric bodies held at different temperatures and connected at a single quantum point contact was found to be related to the local phonon density of states of each reservoir. Similar work has been done involving the phonon dominated thermal transport through more complex connections, such as molecules [9–11].

Here we examine the noise of a thermal current in this limit of weak transmittance, the shot-noise limit. In the same way the granularity of the charge carriers, in say a weak tunneling current, contributes to the current noise, the analogous behavior for phonons should be observed in a thermal current[1]. Experimentally, the ability to detect a single phonon is an ongoing area of interest [12].

In [8] the thermal current between two insulators weakly joined by only a mesoscopic link, modeled as a harmonic spring, was calculated. The actual physical link could be a few chemical bonds or even a small bridge of material, see figure 1. The result of [8] was the thermal analog of the well-known tunneling current formula [7]. In the following sections we examine the intrinsic noise present in such a thermal current. It is assumed the two bodies are only weakly coupled, to lowest

[1] Due to the bosonic nature of phonons, distinguishing the total energy carried by a single phonon or two or more with smaller energy would be difficult to discern.

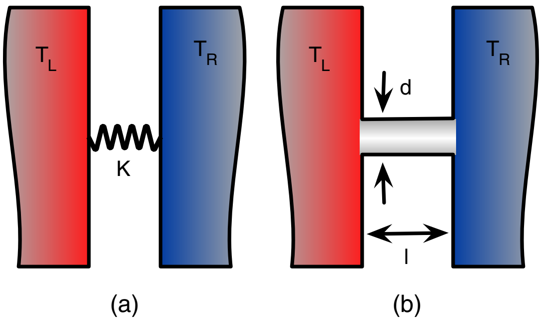

Figure 1. Model of two macroscopic dielectric bodies held at different temperatures T_L and T_R and joined by a single quantum point contact. The contact can be taken to be (a) several atomic bonds with a spring constant K or (b) a small 'neck' of dielectric material of length l and diameter d with an effective spring constant $K = (\pi d^2/4l)Y$, where Y is the Young's modulus of the material.

order in the coupling, this is equivalent to the shot-noise limit of the electronic counterpart.

2. Model and thermal current

We consider the following model, which is illustrated in figure 1: two macroscopic dielectric bodies, labeled left (L) and right (R), act as thermal reservoirs and are held at fixed temperatures T_L and T_R, which generates a thermal gradient. The Hamiltonian for each side is taken, in the harmonic approximation, as ($\hbar = 1$)

$$H_I := \sum_n \omega_{In} a^\dagger_{In} a_{In}, \qquad I = \mathrm{L}, \mathrm{R} \tag{1}$$

where $a^\dagger_{In}$ and a_{In} are phonon creation and annihilation operators for the left and right side, which satisfy

$$[a_{In}, a^\dagger_{I'n'}] = \delta_{nn'}\delta_{II'} \tag{2}$$

and

$$[a_{In}, a_{I'n'}] = [a^\dagger_{In}, a^\dagger_{I'n'}] = 0. \tag{3}$$

The quantum point contact, or weak mechanical link, is modeled as a harmonic potential with spring constant K

$$\delta H := \tfrac{1}{2} K [u^z_{\mathrm{L}}(\mathbf{r}_0) - u^z_{\mathrm{R}}(\mathbf{r}_0)]^2, \tag{4}$$

where $\mathbf{r}_0$ is the point of contact between the two reservoirs and u^z_I is the z-component (the direction normal to each surface) of displacement field $\mathbf{u}_I(\mathbf{r})$. This model of the weak link assumes the compressional strength of the link dominates over other such displacements such as flexural or torsional. In principle these interactions could also be included, this would amount to replacing the spring constant K with a tensor quantity coupling to different components of the phonon field operator.

The displacement field of each reservoir can be expanded in terms of phonon creation and annihilation operators as

$$\mathbf{u}_I(\mathbf{r}) := \sum_n \sqrt{\frac{1}{2\rho\omega_{In}}}[a_{In}\mathbf{f}_{In}(\mathbf{r}) + a^\dagger_{In}\mathbf{f}^*_{In}(\mathbf{r})], \tag{5}$$

where $\mathbf{f}_{In}(\mathbf{r})$ are the normalized vibrational eigenfunctions, and ρ is the mass density.

2.1. Thermal current

Because of energy conservation and using Heisenberg's equation-of-motion, a thermal current operator can be defined as

$$\hat{I}_{\mathrm{th}} := \partial_t H_{\mathrm{R}} = \mathrm{i}[H, H_{\mathrm{R}}], \tag{6}$$

where the full Hamiltonian $H = H_L + H_R + \delta H$. Performing the commutator gives

$$\hat{I}_{\mathrm{th}} = \frac{\mathrm{i}K}{2} \sum_{nn'} \omega_{\mathrm{R}n} \{A_{\mathrm{R}n'} - A_{\mathrm{L}n'}, h_{\mathrm{R}n} a_{\mathrm{R}n} - h^*_{\mathrm{R}n} a^\dagger_{\mathrm{R}n}\}, \tag{7}$$

where $h_{In} := (2\rho\omega_{In})^{-1/2} f^z_{In}$, $A_{In} := h_{In}a_{In} + h^*_{In}a^\dagger_{In}$, and $\{\cdot,\cdot\}$ is the anticommutator. Treating the coupling as the perturbation; within linear response, the thermal current is [8]

$$\langle \hat{I}_{\mathrm{th}} \rangle = 2\pi K^2 \int_0^\infty \mathrm{d}\epsilon\, \epsilon\, N^{zz}_{\mathrm{L}}(\mathbf{r}_0, \epsilon) N^{zz}_{\mathrm{R}}(\mathbf{r}_0, \epsilon)[n^{\mathrm{B}}_{\mathrm{L}}(\epsilon) - n^{\mathrm{B}}_{\mathrm{R}}(\epsilon)], \tag{8}$$

where $n^{\mathrm{B}}_I(\epsilon) = (\exp(\epsilon\beta_{\mathrm{I}}) - 1)^{-1}$ is the Bose distribution with $\beta = (k_B T)^{-1}$, and

$$N^{zz}_I(\mathbf{r}, \omega) = \sum_n |h_{In}(\mathbf{r})|^2 \delta(\omega - \omega_{nI}) \tag{9}$$

is the zz component of the local *spectral* density. It should be noted that equation (9) is *not* equal to the zz component of the local phonon density of states tensor given by [13],

$$g^{ij}_I(\mathbf{r}, \omega) = \sum_n f^i_{In}(\mathbf{r})[f^j_{In}(\mathbf{r})]^* \delta(\omega - \omega_{nI}), \tag{10}$$

but $N^{zz}_I(\mathbf{r}, \omega)$ is related to the imaginary part of the retarded phonon Green's function and is the relevant quantity of interest for the present work. For clarity the superscripts zz will be dropped from here on. Equation (8) is the thermal analog of the expression for an electronic tunneling current, equation (18).

3. Calculation of the phonon shot-noise

Here we calculate the intrinsic noise[2] associated with a thermal current as calculated in section 2.1. The symmetrized noise is defined as [14–16]

$$S_{\mathrm{th}}(\omega) := \tfrac{1}{2} \int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \langle \{\delta\hat{I}_{\mathrm{th}}(t), \delta\hat{I}_{\mathrm{th}}(0)\} \rangle_H, \tag{11}$$

where $\delta\hat{I}_{\mathrm{th}} := \hat{I}_{\mathrm{th}} - \langle \hat{I}_{\mathrm{th}} \rangle_H$.[3] In [17] the short time, or high-frequency ($\omega \to \infty$), noise of a general heat current was studied. Here we investigate the long time or low-frequency ($\omega \to 0$) noise in the weak transmission limit.

To lowest order in the interaction the noise is simply

$$S_{\mathrm{th}}(\omega) := \tfrac{1}{2} \int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \langle \{\hat{I}_{\mathrm{th}}(t), \hat{I}_{\mathrm{th}}(0)\} \rangle_{H_0}, \tag{12}$$

[2] The noise generated by the system of study and not from external experimental equipment.

[3] Sometimes the factor of 1/2 is omitted and thus will change some subsequent formulas by a factor of two.

where $H_0 := H_L + H_R$,

$$\langle \hat{O} \rangle_{H_0} = \frac{\mathrm{Tr}\ \mathrm{e}^{-\beta H_0} \hat{O}}{\mathrm{Tr}\ \mathrm{e}^{-\beta H_0}}, \tag{13}$$

and

$$\hat{O}(t) = \mathrm{e}^{\mathrm{i}H_0 t} \hat{O}\, \mathrm{e}^{-\mathrm{i}H_0 t}. \tag{14}$$

Using equation (7), along with dropping anomalous terms, the zero-frequency component of the noise is[4]

$$S_{\mathrm{th}}(\omega = 0) = 2\pi K^2 \int_0^\infty \mathrm{d}\epsilon\, \epsilon^2 N_{\mathrm{L}}(\epsilon) N_{\mathrm{R}}(\epsilon) \times \left\{ n_{\mathrm{L}}^{\mathrm{B}}(\epsilon)[1 + n_{\mathrm{R}}^{\mathrm{B}}(\epsilon)] + n_{\mathrm{R}}^{\mathrm{B}}(\epsilon)[1 + n_{\mathrm{L}}^{\mathrm{B}}(\epsilon)] \right\} \tag{15}$$

or

$$S_{\mathrm{th}}(\omega = 0) = 2\pi K^2 \int_0^\infty \mathrm{d}\epsilon\, \epsilon^2 \coth\left[\frac{\epsilon}{2k_{\mathrm{B}}}\left(\frac{1}{T_{\mathrm{R}}} - \frac{1}{T_{\mathrm{L}}}\right)\right] \times N_{\mathrm{L}}(\epsilon) N_{\mathrm{R}}(\epsilon)[n_{\mathrm{L}}^{\mathrm{B}}(\epsilon) - n_{\mathrm{R}}^{\mathrm{B}}(\epsilon)]. \tag{16}$$

It is illustrative to compare equation (16) to the electronic expression of the zero-frequency component of the shot-noise,

$$S_{\mathrm{el}}(\omega = 0) = e\langle \hat{I}_{\mathrm{el}}(eV)\rangle \coth(eV\beta/2), \tag{17}$$

where for a tunneling current

$$\langle \hat{I}_{\mathrm{el}}(eV)\rangle = 2\pi e |T|^2 \sum_\sigma \int \mathrm{d}\omega\, \rho_{\mathrm{L}}(\mathbf{r}\sigma, \omega - eV) \times \rho_{\mathrm{R}}(\mathbf{r}\sigma, \omega)[n_{\mathrm{L}}^{\mathrm{F}}(\omega - eV) - n_{\mathrm{R}}^{\mathrm{F}}(\omega)]. \tag{18}$$

Here $|T|^2$ is the transmission probability, $\rho_I(\omega)$ is the electronic local density of states, and $n_I^{\mathrm{F}}(\omega)$ is the Fermi distribution function.

Assuming, as in most cases of interest, the phonon spectral density obeys a power-law at low energies, $N_I(\omega) \propto \omega^\alpha$ and letting $T_{\mathrm{R}} \to 0$ for simplicity, the temperature dependence of the noise is given as

$$S_{\mathrm{th}}(\omega = 0) \propto T^{3+2\alpha}. \tag{19}$$

3.1. Equilibrium noise

In the limit $T_{\mathrm{L}} \to T_{\mathrm{R}}$ there is no net heat current; nonetheless, there remains thermal fluctuations given by

$$S_{\mathrm{th}}(\omega = 0) = 2k_{\mathrm{B}} T^2 G_{\mathrm{th}}, \tag{20}$$

where

$$G_{\mathrm{th}} := \lim_{T_{\mathrm{L}} \to T_{\mathrm{R}}} \frac{I_{\mathrm{th}}}{T_{\mathrm{L}} - T_{\mathrm{R}}} = 2\pi K^2 \int_0^\infty \mathrm{d}\epsilon\, \epsilon\, N_{\mathrm{L}}(\epsilon) N_{\mathrm{R}}(\epsilon) \frac{\partial n^{\mathrm{B}}(\epsilon)}{\partial T} \tag{21}$$

is the linear thermal conductance. This is the phonon analog of Nyquist–Johnson noise. In an electronic system the Nyquist–Johnson noise is given by

$$S_{\mathrm{el}}(\omega = 0) = 2k_{\mathrm{B}} T G_{\mathrm{el}}. \tag{22}$$

Because the thermal noise is a measure of energy fluctuations, and not charge, an additional factor of temperature, T, appears in equation (20), as compared to equation (22). It should also be noted that, in equilibrium the formal relationship between the noise and conductivity, equation (20), is independent of the model used here, and is a consequence of the fluctuation-dissipation theorem.

[4] Because the phonons are noninteracting, in the harmonic approximation used here, the correlation functions involved can easily be evaluated.

3.2. Fano factor

The Fano factor F, or noise-to-signal ratio, can also be of interest. In the case of charge shot-noise, from equation (17) and in the low temperature limit, $F_{\mathrm{el}} := S_{\mathrm{el}}/I_{\mathrm{el}} = e$, the charge of the charge carrier. This has been used to measure the fractional charge, e.g. $e/3$, $e/5$, of the quasiparticles predicted for a quantum Hall fluid [18–21].

Here we determine the Fano factor for a thermal current. To simplify things let $T_{\mathrm{R}} \to 0$, thus

$$F_{\mathrm{th}} := \frac{S_{\mathrm{th}}}{I_{\mathrm{th}}} = \frac{\int_0^\infty \mathrm{d}\epsilon\, \epsilon^2 N_{\mathrm{L}}(\epsilon) N_{\mathrm{R}}(\epsilon) n_{\mathrm{L}}^{\mathrm{B}}(\epsilon)}{\int_0^\infty \mathrm{d}\epsilon\, \epsilon\, N_{\mathrm{L}}(\epsilon) N_{\mathrm{R}}(\epsilon) n_{\mathrm{L}}^{\mathrm{B}}(\epsilon)}. \tag{23}$$

Again assuming a power-law form of the phonon spectral density and re-scaling the integrals by letting $x = \epsilon\beta$ gives

$$F_{\mathrm{th}} = \frac{\int_0^\infty \mathrm{d}x\, x^{2+2\alpha}[\mathrm{e}^x - 1]^{-1}}{\int_0^\infty \mathrm{d}x\, x^{1+2\alpha}[\mathrm{e}^x - 1]^{-1}} k_{\mathrm{B}} T := C(\alpha) k_{\mathrm{B}} T. \tag{24}$$

Thus the Fano factor is not independent of specific details of the system, as in the electronic case, but is independent of all material parameters and only depends on the energy dependence of the spectral density. For planar surfaces [8, 13] $\alpha = 1$ and the integrals can be done analytically giving

$$F_{\mathrm{th}} = C(1) k_{\mathrm{B}} T = \frac{360\, \zeta(5)}{\pi^4} k_{\mathrm{B}} T \approx 3.83\, k_{\mathrm{B}} T, \tag{25}$$

where $\zeta(x)$ is the Riemann–Zeta function. One could loosely interpret equation (25) as the average energy of the transmitted phonons through the weak link.

4. Application: nanometer-scale silicon link

Here we apply the results of the previous sections to calculate both the thermal current and the thermal noise for the following realistic, but simple, model. We assume the weak link is a cylindrical bridge made of silicon (Si) with a length $l = 10$ nm and diameter $d = 1$ nm. This link connects two semi-infinite Si reservoirs, see figure 1.

The low-energy (much smaller than the Debye energy) phonon spectral density of Si at a planar surface has been calculated in [8, 13] and is

$$N(\epsilon) = C\epsilon, \qquad C \approx 1.3 \times 10^8\ \mathrm{cm}^2\ \mathrm{erg}^{-2}. \tag{26}$$

The longitudinal stiffness, or effective spring constant K, of the mechanical link can be approximated by using the bulk Young's modulus Y of Si such that

$$K = \frac{\pi d^2}{4l} Y. \tag{27}$$

For Si, $Y \approx 1.3 \times 10^{12}$ dyn cm^{-2}; therefore, with the given dimensions of the link,

$$K \approx 1.0 \times 10^4\ \mathrm{erg\ cm}^{-2}. \tag{28}$$

With (26) and (28) the thermal current equation (8) and the noise equation (16) can now be found within this model,

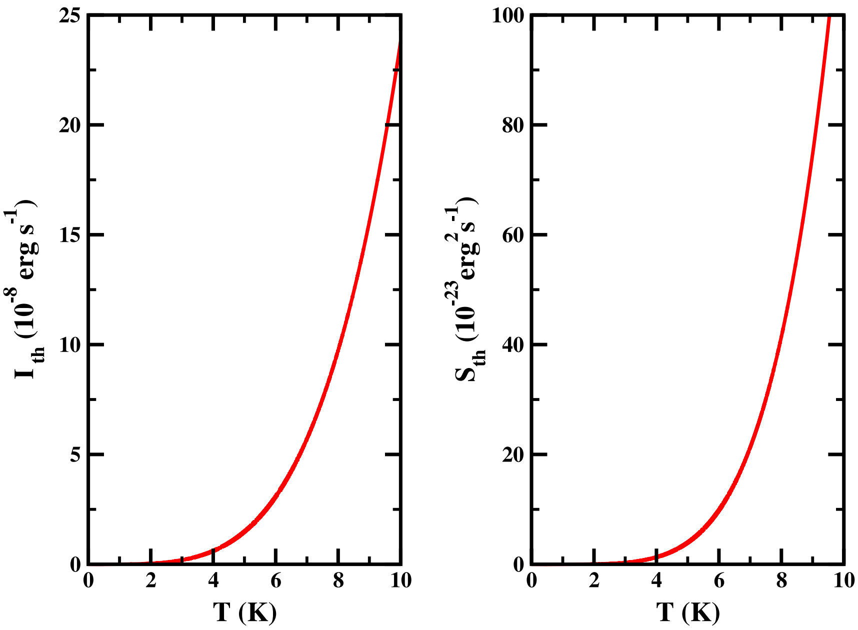

Figure 2. The thermal current (left) and thermal noise (right) between two dielectric bodies held at a temperature difference T and connected by a nanometer-scale mechanical link. Silicon is used to model both the thermal reservoirs and the link, see section 4.

see figure 2. Although measuring the thermal current of such a system is experimentally feasible, as can be seen, the noise, or fluctuations about the mean, is many orders of magnitude smaller than the average current and is probably covered up by the thermal fluctuations of the environment and currently unmeasurable.

5. Discussion

Besides the experimental ability to detect single phonons, and thus the phonon shot-noise, further conditions are needed to be in the shot-noise regime. Within the model considered here, the temperature must remain well below any resonant modes of the weak link, also the link should remain in the mesoscopic regime, i.e. smaller than the phonon coherence length, which in itself depends on temperature. This would suggest an upper bound on temperatures of roughly 10 K.

Of course phonon noise is not only of interest for the work presented here, but could also be used to study other behavior, such as demonstrating phonon bunching in a phonon Hanbury-Brown and Twiss experiment [22].

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